PISOT NUMBERS AND THE WIDTH OF MEROMORPHIC FUNCTIONS

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ABSTRACT

Associated with the set S of Pisot numbers is a set $\mathcal{C}$ of rational functions which are bounded in modulus by 1 on the unit circle, and have a single pole inside the circle at the reciprocal of a Pisot number. Dufresnoy and Pisot introduced an algorithm, based on Schur's classical algorithm, which can be used to generate all members of $\mathcal{C}$, and hence all of S. This paper can be regarded as the theoretical preparation for a future paper which the author is currently writing, which will deal with applications of this algorithm to questions about Pisot numbers, Pisot sequences and Salem numbers.

Each f in $\mathcal{C}$ is determined by an infinite path in a tree associated with the above algorithm. We introduce a functional on $\mathcal{C}$, $w(f)$, called the "width" of f, which has a natural interpretation in terms of this tree. From $w(f)$, we obtain a function $w(\theta)$ on S which is the maximum of $w(f)$ over all f with poles at θ^{-1}. The quantity $w(\theta)$ measures the complexity of S near the point θ. For example $w(\theta) = 0$ if θ is an isolated point of S, and $w(\theta) \geq h$ if θ is in $S^{(h)}$, the hth derived set of S.

Our main result is an explicit formula for $w(f)$. A simple corollary of this formula is that $\min S^{(h)} \geq (h+1)^{1/2}$. The sharpness of this inequality is investigated by special arguments. Since $w(f)$ is not continuous, a formula developed for $\lim w(f_n)$ is of some interest.

FOOTNOTES

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PISOT NUMBERS AND THE WIDTH OF MEROMORPHIC FUNCTIONS

By David W. Boyd (1)

Introduction: Our main purpose here is to study certain aspects of the structure of the set S of Pisot (or Pisot-Vijayaraghavan) numbers. In particular, we shall be interested in the successive derived sets $S^{(h)}$ of S. The reader will recall that S is the set of real algebraic integers $\theta > 1$ all of whose remaining conjugates lie strictly within the unit circle. The set S is closed, according to a well-known result of Salem [14].

Dufresnoy and Pisot [4-7] have shown that an effective way to determine detailed properties of S is to relate questions concerning S to questions about a set $\mathcal{C}$ of rational functions bounded by 1 in modulus on the unit circle. In [6] they solved the "coefficient problem" for $\mathcal{C}$ by giving a sequence of inequalities $w_n(u_0,\ldots,u_{n-1}) \le u_n \le w_n^*(u_0,\ldots,u_{n-1})$ which must be satisfied by the sequence $\{u_n\}$ of Taylor coefficients of any f in $\mathcal{C}$. Using this result they were able to determine all the numbers in $S \cap [1,\tau]$, and show that $\tau = (\sqrt{5}+1)/2 = \min S^{(1)}$.

Our main tool will be a functional $w(f)$ defined for f in $\mathcal{C}$ by $\lim (w_n^* - w_n)/2$. We call $w(f)$ the width of f. This limit was shown to exist in [6], but no explicit use was made of this fact. In section 1, we give a combinatorial interpretation of $w(f)$ which suggests why it is an interesting object of study.

A basic result is Theorem 1, which gives an explicit formula for $w(f)$ in terms of the geometric mean of $1 - |f|^2$ on the unit circle. This formula is valid for the class of meromorphic functions considered by

Chamfy [3] and described in section 2. As a Corollary of Theorem 1, we show that $\min S^{(h)} \geq (h+1)^{1/2}$ which improves the estimate $h^{1/4}$ of [5].

The functional w is not continuous on $\mathcal{C}$, but it is possible to give some information about $w(f_n)$ if $f_n \to f$ in $\mathcal{C}$ and we investigate this in section 4. Corollary 7 of Theorem 2 gives a formula for $\lim w(f_n)$ if $f_n = (A + z^n B)/(Q + z^n C)$. We use this result to investigate $w(f)$ on the set of f associated with the set $S \cap [1,2)$ and show that $w(f)$ attains a maximum $w(f) = 1.9877\ldots$ for $f = (1-z)(1+z^5)/(1-2z+z^5-z^6)$. Another application settles a natural question concerning the relationship between $w(f)$ and the "index of derivability" of f.

An interesting (and open) question is whether or not our estimate of $\min S^{(h)}$ is sharp in the sense that $\min S^{(h)} \sim h^{1/2}$. In section 5, Theorem 3, we show that if $\theta = k$, a rational integer, then $k \in S^{(N_k)}$, where $N_k/k \to 2/(2\sqrt{2} - 1) > 1$. This shows that $\liminf(\min S^{(h)}/h) > 1$. We also give a special argument to show $3 \in S^{(3)}$. The results of section 2 do not rule out the possibility that $3 \in S^{(7)}$.

1. The class $\mathcal{C}$ and the algorithm of Dufresnoy and Pisot.

Associated with each θ in S is its minimal polynomial $P(z)$ of degree s, say, and the reciprocal polynomial $Q(z) = z^s P(z^{-1})$ which has $Q(0) = 1$, integer coefficients, and exactly one zero θ^{-1} in $|z| \leq 1$. If $A(z)$ is a polynomial with integer coefficients, not identical with Q, having $A(0) > 0$ and $|A(z)| \leq |Q(z)|$ for $|z| = 1$, then the rational function $f = A/Q$ is said to be associated with θ. Such A do exist, in fact $A = \pm P$ is suitable unless $Q(z) = 1 - qz + z^2$ in which case $A(z) = 1$ is available.

The set of such f is denoted $\mathcal{C}$. It is clear that f has the following properties:

(i) f is holomorphic in $|z| \leq 1$ except for a simple pole at $z = \theta^{-1} < 1$,

(ii) $|f(z)| \leq 1$ on $|z| = 1$,

(iii) $f(z) = u_0 + u_1 z + \ldots$ for $|z| < \theta^{-1}$, where the u_n are integers and $u_0 \geq 1$.

In fact $\mathcal{C}$ is characterized by these properties by a result of Pisot (see Salem [15]).

The set $\mathcal{C}$ can be given a topology by defining convergence to mean uniform convergence (in the metric of the Riemann sphere) on compact subsets of $|z| < 1$. A basic result of Pisot [13,p.42] is that, for any $\delta > 0$, the set $\{f \in \mathcal{C} : \theta \leq \delta^{-1}\}$ is compact. This implies that S is closed. If $f_n \to f$ in $\mathcal{C}$ then the Taylor expansion of f_n coincides with that of f to an arbitrarily large number of terms.

The derived sets $\mathcal{C}^{(h)}$ of $\mathcal{C}$ have been characterized and used to characterize the sets $S^{(h)}$. Dufresnoy and Pisot [4] showed that $f \in \mathcal{C}^{(1)}$ if and only if $f = A/Q$ where $|A(z)| \leq |Q(z)|$ for $|z| = 1$ with equality at a finite number of points at most. Thus f is an isolated point of $\mathcal{C}$ if and only if $f(z) = \pm z^s Q(z^{-1})/Q(z)$. Grandet-Hugot [9] has characterized $\mathcal{C}^{(h)}$ for all h. The result for $h = 2$ is that $A/Q \in \mathcal{C}^{(2)}$ if and only if there are polynomials B and C with integer coefficients, not both identically zero, such that B/Q and C/Q are in $\mathcal{C}^{(1)}$ or identically zero, and $(A + z^nB)/(Q + z^nC)$ is in $\mathcal{C}^{(1)}$ for all $n \geq 0$. In general, the existence of $2^h - 2$ auxiliary polynomials is required. In addition, the Pisot number θ is in $S^{(h)}$ if and only if there is an $f \in \mathcal{C}^{(h)}$ which is associated with θ [4,9]. It is shown in [5] that $\min S^{(h)} \to \infty$, so the following make sense:

Definition 1. If $f \in \mathcal{C}$, define $h(f) = \max \{h: f \in \mathcal{C}^{(h)}\}$. If $\theta \in S$, define $h(\theta) = \max \{h: \theta \in S^{(h)}\}$. We call $h(f)$ (or $h(\theta)$), the index of derivability of f (or θ respectively).

To investigate $\mathcal{C}$, Dufresnoy and Pisot [6] investigated the coefficient problem for a wider class of functions, namely the set of f satisfying (i) and (ii), but with real coefficients and $f(0) = u_0 \geq 1$. They showed that the coefficient sequence u_n for such an f satisfies the following system of inequalities:

$$(1) \qquad \begin{aligned} 1 &\leq u_0 \\ u_0^2 - 1 &\leq u_1 \\ w_n(u_0,\dots,u_{n-1}) \leq u_n &\leq w_n^*(u_0,\dots,u_{n-1}) \quad , \quad n = 2,3,\dots \end{aligned}$$

The bounds on u_n, $n \geq 2$ restrict u_n to a finite interval except in one case, namely if $u_0 = 1$ and $n = 2$, then $w_2^* = \infty$. The functions w_n, w_n^* are rational functions of their arguments. They are recursively generated by means of certain polynomials D_n, D_n^* as described in [6]. The system (1) in fact characterizes the coefficient sequences for a larger class of functions than they considered, as we point out in Section 2.

If $u_s = w_s$ or $u_s = w_s^*$ for any s, then f is uniquely determined and in fact is of the form $z^s Q(z^{-1})/Q(z)$ for a polynomial Q (which need not have integer coefficients). In this case $w_n = w_n^*$ for all $n > s$. If $w_n \neq w_n^*$ then

(2) $$w_{n+1}^* - w_{n+1} = 4(u_n - w_n)(w_n^* - u_n)/(w_n^* - w_n)$$

from which one deduces that $\{w_n^* - w_n\}$ is a decreasing sequence.

<u>Definition 2</u>. We define the <u>width</u> of $f \in \mathcal{C}$ by

$$w(f) = \lim_{n \to \infty} (w_n^*(u_0,\ldots,u_{n-1}) - w_n(u_0,\ldots,u_{n-1}))/2 .$$

We can rewrite (2) in terms of $a_n = u_n - w_n$ and $b_n = w_n^* - u_n$. Then

(3) $$(a_{n+1} + b_{n+1})/2 = a_n b_n/((a_n + b_n)/2) .$$

A result of [6,p.83] becomes

(4) $$\sum_{n=3}^{\infty} (a_n - b_n)^2 < \infty ,$$

so that

(5) $$\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = \lim_{n \to \infty} (a_n + b_n)/2 = w(f) .$$

For the class $\mathcal{C}$, there is a relation between $h(f)$ and $w(f)$ which follows from a result of Grandet-Hugot:

<u>Lemma 1</u>. If $f \in \mathcal{C}$ then $h(f) \leq w(f)$.

<u>Proof</u>: A result of [9,p.24] shows that if $f \in \mathcal{C}^{(h)}$, then for all sufficiently large n, $w_n + h \leq u_n \leq w_n^* - h$. Thus $2h \leq \lim(w_n^* - w_n) = 2w(f)$.

For the class $\mathcal{C}$, the fact that the u_n are integers means that (1) completely characterizes the coefficient sequences. Hence (1) provides an (unbounded) algorithm for the determination of all $f \in \mathcal{C}$. In a later paper we shall show how this algorithm can be used to determine all Pisot numbers in certain intervals and to give lower bounds on certain classes of Salem numbers, using results of [2]. Such an algorithm is most naturally described in terms of a tree $\mathcal{J}$. Each vertex of this tree at height n is specified by a finite sequence of integers $(u_0,\ldots,u_n)$ satisfying (1), where the empty sequence $*$ is the root of the tree at height -1. The successors of $(u_0,\ldots,u_n)$ are all $(u_0,\ldots,u_n,u_{n+1})$ with $w_{n+1} \le u_{n+1} \le w^*_{n+1}$ (if any such exist). Except for vertices of the form $*$, (u_0) or $(1,u_1)$, all other vertices have a finite number of successors, i.e. have finite valence.

An $f \in \mathcal{C}$ with coefficient sequence $\{u_n\}$ corresponds to a path to infinity $*$, (u_0), $(u_0,u_1),\ldots$ in $\mathcal{J}$. The quantities $[a_n]$, $[b_n]$ are the number of paths which branch to the left and right respectively at level n. Thus (5) expresses a certain homogeneity of the tree and makes apparent the intrinsic signifigance of $w(f)$.

We can show the fruitfulness of this point of view by the following result which is a simple application of König's infinity lemma [12,p.381]. We first give

Definition 3. If $\theta \in S$, define $w(\theta) = \sup\{w(f) : f \in \mathcal{C}$ and f has a pole at $\theta^{-1}\}$.

Lemma 2. The supremum in Definition 1 is a maximum. That is, there is an $f \in \mathcal{C}$ with pole at θ^{-1} for which $w(f) = w(\theta)$.

Proof: Let the set of f associated with θ be denoted $\mathcal{C}_\theta$. Consider the subtree $\mathcal{J}_\theta$ of $\mathcal{J}$ corresponding to expansions of f in $\mathcal{C}_\theta$. Every vertex in $\mathcal{T}_\theta$ has a finite valence. For, as in [17], we have

$$u_0^2 + (u_1 - u_0\theta)^2 + (u_2 - u_1\theta)^2 + \ldots \leq 1 + \theta^2 ,$$

by applying Bessel's inequality to $(1 - \theta z)f(z)$. Hence, by induction, $(u_0,\ldots,u_n)$ has only a finite number of possible successors for any n.

For any $f = u_0 + u_1 z + \ldots$, consider $g = v_0 + v_1 z + \ldots$ in with $v_j = u_j$ for $j < n$, but $v_n - u_n = c \neq 0$. Write a_j' , b_j' for $a_j(v_0,\ldots,v_{j-1})$ and $b_j(v_0,\ldots,v_{j-1})$. Then (3) and (5) imply

(6) $$w(g) \leq (a_{n+1}' + b_{n+1}')/2 = (a_n + c)(b_n - c)/((a_n + b_n)/2) \to w(f) - k^2/w(f),$$

as $n \to \infty$. Thus, there is an integer $N(f)$ such that if $n \geq N(f)$, and $g \neq f$ has its first n components equal to those of f then $w(f) > w(g)$.

We replace $\mathcal{T}_\theta$ by a tree $\mathcal{T}_\theta'$ obtained by "pruning" f at level $N(f)$. That is, all vertices $(u_0,\ldots,u_{N(f)},\ldots,u_m)$ are replaced by the single vertex $(u_0,\ldots,u_{N(f)})$ which is labelled f. The resulting tree has no paths to infinity and all vertices have finite valence, hence $\mathcal{T}_\theta'$ is a finite tree, by König's lemma. Each terminal vertex of $\mathcal{T}_\theta'$ is associated with an f which maximizes $w(f)$ over all g with initial coefficients $(u_0,\ldots,u_{N(f)})$. One of these f necessarily maximizes $w(f)$ over $\mathcal{C}_\theta$.

2. <u>The generalized Schur algorithm</u>. Theorem 1 of the next section applies to a class of meromorphic functions which contains $\mathcal{C}$. We shall let $\mathcal{M}_p$ denote the class of functions which are meromorphic in $|z| < 1$, holomorphic at the origin, have a finite number p of poles in $|z| < 1$, say at $z = \theta_1^{-1}$, $\theta_2^{-1}, \ldots, \theta_p^{-1}$ and are such that

$$g(z) = f(z)(z - \bar{\theta}_1)^{-1}(1 - \theta_1 z)\ldots(z - \bar{\theta}_p)^{-1}(1 - \theta_p z)$$

is bounded by 1 in modulus in $|z| < 1$. Then g has radial limits a.e. [11,p.38], so the definition of f extends to $|z| = 1$, and $|f(z)| \leq 1$ for almost all such z.

Schur [16] showed that the set $\mathcal{M}_0$ could be studied by applying the following familiar process: set $f_0(z) = f(z)$, and then successively

$$(7) \qquad f_{n+1}(z) = (f_n(z) - \gamma_n)/z(1 - \bar{\gamma}_n f_n(z)), \text{ where } \quad \gamma_n = f_n(0).$$

If f_n is a non-constant function in $\mathcal{M}_0$ then f_{n+1} is in $\mathcal{M}_0$. If f_s is a constant γ_s with $|\gamma_s| < 1$ then f_n is identically 0 for $n > s$. If f_s is a constant of modulus 1, then f_n is undefined for $n > s$. In this latter case f is a rational function of the form $\varepsilon z^s \bar{Q}(z^{-1})/Q(z)$, where Q is a polynomial with no zeros in $|z| \leq 1$ and $|\varepsilon| = 1$.

The coefficient γ_n is a function $\Phi_n(u_0,\ldots,u_n)$. Schur proved that the inequalities $|\Phi_n(u_0,\ldots,u_n)| \leq 1$, with strict inequality for all n or else with equality for $n = s$, characterize the coefficients of $f \in \mathcal{M}_0$.

Chamfy [3] showed how to extend these considerations to the class $\mathcal{M}_p$. Now, in addition to the basic transformation (7), which applies if $|f_n(0)| < 1$, there are two other transformations required if $|f_n(0)| > 1$ or $= 1$, respectively. If $f_n = v_0 + v_m z^m + \ldots$ with $|v_0| > 1$ and $v_m \neq 0$, then one passes from f_n to f_{n+m}, while if $|v_0| = 1$ and $v_m \neq 0$, one passes from f_n to f_{n+2m}. Thus a sequence

(8) $$f_0, f_1, \ldots, f_{n_0}, \ldots, f_s, \ldots$$

is generated, where certain of $f_1, \ldots, f_{n_0-1}$ may be undefined. However, Chamfy showed that, for some finite n_0, f_{n_0} is holomorphic and the process eventually becomes (7), so Schur's results apply.

Chamfy also showed how to form the inequalities analogous to (1) in case f has real coefficients. Grandet-Hugot [9] treated the case of complex coefficients, and we give a partial account of Theorem 1 of [9] as Lemma 3. As with Schur's original process, the sequence may eventually end with a constant of modulus 1, f_s. We shall say f has range s in this case, and range ∞ otherwise. An f of range s is of the form $\varepsilon z^s \overline{Q}(z^{-1})/Q(z)$, where Q has p zeros in $|z| < 1$, and $|\varepsilon| = 1$.

<u>Lemma 3.</u> Let f be in $\mathcal{M}_p$ with range s (possibly ∞), and expansion $u_0 + u_1 z + \ldots$ near 0. There is an n_0 such that, for $n_0 \leq n \leq s$, a unique pair of polynomials A_n and Q_n of degree at most n exists which satisfy:

(9) $Q_n(0) = 1$ and Q_n has p zeros in $|z| < 1$,

(10) $|A_n(z)| < |Q_n(z)|$ for $|z| = 1$, if $n < s$, while

$|A_s(z)| = |Q_s(z)|$ for $|z| = 1$,

(11) $A_n(z)/Q_n(z) = u_0 + u_1 z + \ldots + u_n z^n + s_{n+1} z^{n+1} + \ldots$

(12) $Q_n(z)\overline{Q}_n(z^{-1}) - A_n(z)\overline{A}_n(z^{-1}) = \omega_n$, independent of z,

(13) $$|u_{n+1} - s_{n+1}| \leq \omega_n \ .$$

If A_n has leading coefficient γ_n, then

(14) $$\gamma_{n+1} = (u_{n+1} - s_{n+1})/\omega_n \ ,$$

(15) $$\omega_{n+1} = \omega_n(1 - |\gamma_{n+1}|^2) \ .$$

Definition 4. For $f \in \mathcal{M}_p$ we define $w(f) = \lim_{n \to \infty} \omega_n$ (= 0 if f has range $s < \infty$).

Note that (15) shows ω_n is non-increasing, and that (13) corresponds to (1) so Definitions 1 and 4 are consistent. We note that if f_n is the sequence of (2) then, for $n_0 \leq n \leq s$, we have

(16) $$f(z) = (A_n(z) + z^{n+1} \bar{Q}_n(z^{-1})f_{n+1}(z))/(Q_n(z) + z^{n+1} \bar{A}_n(z^{-1})f_{n+1}(z)),$$

and that $f_n(0) = \gamma_n$ is the sequence of Lemma 3. The sequences A_n, Q_n satisfy recurrence relations given in [9], although it should be pointed out that (4) and (5) of [9,p.5] are incorrect. The correct relations can be derived from (7) and (16). We note that, if $f \in \mathcal{M}_0$, then γ_n is defined for all $n \leq s$, and

(17) $$\omega_n = \prod_{k=0}^{n} (1 - |\gamma_k|^2) \ .$$

It is worth mentioning that Wall [18] has recast Schur's algorithm (for $\mathcal{M}_0$) as a continued fraction algorithm in which A_n/Q_n appears as a convergent of even order.

The reader familiar with [6] should note that $D_n(z) = A_{n-1}(z) - z^n Q_{n-1}(z^{-1})$, and $D_n^*(z) = A_{n-1}(z) + z^n Q_{n-1}(z^{-1})$.

Remark. Chamfy [3] in fact formulates her results for the class of functions which are meromorphic in the closed disk $|z| \leq 1$, and satisfy the other conditions in the description of $\mathcal{M}_p$. The conditions derived by Schur [16], on the other hand, are necessary and sufficient to characterize the coefficient sequence of an $f \in \mathcal{M}_0$, so in fact Chamfy's results give only necessary conditions for the class she considers. However, it is easy to verify that the considerations of [3] carry through for $f \in \mathcal{M}_p$; one need only verify that the Lemma on p.218 of [3] is valid for this class of f, and this is an easy matter.

Dufresnoy and Pisot [6] do not discuss the question of which class of functions (1) characterizes. For their purposes, it is enough that (1) is a necessary condition for f to be in $\mathcal{C}$. In fact, (1) is both necessary and sufficient for f, with real coefficients and $u_0 > 1$ or $u_0 = 1$, to be in $\mathcal{M}_1$. If such an f has integer coefficients then it is rational, by [15], and hence is in $\mathcal{C}$. Thus (1) provides a complete characterization in this case.

3. A formula for the width of a function.

We shall need some properties of the geometric mean of a bounded measurable function defined on $|z| = 1$. If $g(e^{it}) \geq 0$, we write

$$\mathcal{G}(g) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log g(e^{it})\, dt\right\}.$$

We will always have $\mathcal{L}(g) < \infty$ but possibly $\mathcal{L}(g) = 0$ if $\log(g)$ is not integrable. Some well-known properties of $\mathcal{L}$ [10,pp.136-138] are

$$\mathcal{L}(g_1 g_2) = \mathcal{L}(g_1)\mathcal{L}(g_2) \tag{18}$$

$$\mathcal{L}(g_1 + g_2) \geq \mathcal{L}(g_1) + \mathcal{L}(g_2) \tag{19}$$

$$\mathcal{L}(g) \leq \mathcal{A}(g) = (2\pi)^{-1} \int_0^{2\pi} g(e^{it})\, dt \; . \tag{20}$$

Also, we have Jensen's formula [11,p.68]: if $f \in H^1$, $f(0) \neq 0$, and f vanishes in $|z| < 1$ exactly at $\alpha_1, \ldots, \alpha_n$, then

$$\mathcal{L}(|f|) = \lambda\, |f(0)|\,|\alpha_1|^{-1} \ldots |\alpha_m|^{-1} \quad , \tag{21}$$

where $\lambda \geq 1$ is a contribution due to the singular part of f. If f is holomorphic in $|z| \leq 1$ then $\lambda = 1$. Finally, we will need a form of Szegö's theorem [11,p.50]: if $g(e^{it}) \geq 0$ is integrable on $[0,2\pi)$, then

$$\mathcal{L}(g) = \inf_P \mathcal{A}(|P|^2 g) \quad , \tag{22}$$

where $P(z) = 1 + b_1 z + \ldots + b_k z^k$.

<u>Theorem 1</u>. Let f be in $\mathcal{M}_p$ with poles at $\theta_1^{-1}, \ldots, \theta_p^{-1}$. Then

$$w(f) = \theta_1^2 \ldots \theta_p^2\, \mathcal{L}(1 - |f|^2) \quad . \tag{23}$$

Furthermore, if $f = A/Q$ is a rational function, and if $\Omega(z) = z^r(Q(z)\bar{Q}(z^{-1}) - A(z)\bar{A}(z^{-1}))$, where r is chosen so that Ω is a polynomial with $\Omega(0) \neq 0$, then

(24) $$w(f) = |Q(0)|^{-2}\, \mathcal{L}(|\Omega|) \quad ,$$

(25) $$w(f) = |Q(0)|^{-2}\, |\Omega(0)| \prod_{\alpha} \max(|\alpha|,1) \quad ,$$

where the product in (25) is over all roots of Ω .

Proof: We begin by showing that we can reduce the proof to the case of holomorphic f. Write $A_n^*(z) = z^n \overline{A}_n(z^{-1})$ and $Q_n^*(z) = z^n\, \overline{Q}_n(z^{-1})$, and then, for $n \geq n_0$, (16) is $f = (A_n + zQ_n^* f_{n+1})/(Q_n + zA_n^* f_{n+1})$. Since the numerator and denominator are holomorphic, the denominator, D, must vanish at the poles of f. Using (12), we have, for $|z| = 1$,

$$1 - |f|^2 = (|Q_n|^2 - |A_n|^2)(1 - |f_{n+1}|^2)/|D|^2 = \omega_n(1 - |f_{n+1}|^2)/|D|^2 \ .$$

Thus, from (18) and (21)

(26) $$\mathcal{L}(1 - |f|^2) = \omega_n \mathcal{L}(1 - |f_{n+1}|^2)\, \mathcal{L}(|D|)^{-2}$$

$$\leq \omega_n\, \mathcal{L}(1 - |f_{n+1}|^2)\, \theta_1^{-2} \cdots \theta_p^{-2} \quad ,$$

since $D(0) = 1$ and it vanishes at $\theta_1^{-1}, \ldots, \theta_p^{-1}$.

Now, inverting (16), we have $f_{n+1} = (fQ_n - A_n)/z(Q_n^* - A_n^* f)$. At $z = 0$, using (11) and (12), the expansion of the denominator of this expression begins $\omega_n z^{n+1}$. For $|z| = 1$, we have

(27) $$1 - |f_{n+1}|^2 = (|Q_n|^2 - |A_n|^2)(1 - |f|^2)/|Q_n^* - A_n^* f|^2$$

$$= \omega_n\, (1 - |f|^2)/|Q_n^* - A_n^* f|^2 \ .$$

If B is the product $(1 - \theta_1 z)(\overline{\theta}_1 - z)^{-1} \cdots (1 - \theta_p z)(\overline{\theta}_p - z)^{-1}$ so that

$g = Bf$ is holomorphic in $|z| < 1$, we have, since $|B| = 1$ on $|z| = 1$,

$$(28) \quad \mathcal{L}(|Q_n^* - A_n^* f|) = \mathcal{L}(|BQ_n^* - A_n^* Bf|) \geq |B(0)|\omega_n = \theta_1^{-1} \cdots \theta_p^{-1} \omega_n .$$

Thus, (27) and (28) give

$$(29) \quad \mathcal{L}(1 - |f_{n+1}|^2) \leq \omega_n \mathcal{L}(1 - |f|^2)\, \omega_n^{-2}\, \theta_1^2 \cdots \theta_p^2 .$$

Combining (26) and (29) we see that

$$(30) \quad \theta_1^2 \cdots \theta_p^2 \mathcal{L}(1 - |f|^2) = \omega_n \mathcal{L}(1 - |f_{n+1}|^2) , \quad n \geq n_0 .$$

Now (17) and (15) together imply that

$$(31) \quad w(f) = \omega_n \lim_{m \to \infty} \prod_{j = n+1}^{m} (1 - |\gamma_j|^2) = \omega_n w(f_{n+1}) .$$

Comparing (30) and (31), we see that (23) will follow for f as soon as it is proved for f_{n+1}, which is holomorphic. Thus, we now assume that f is holomorphic. For such f, (26) (or (30)) gives $\mathcal{L}(1 - |f|^2) \leq \omega_n$, and letting $n \to \infty$, $\mathcal{L}(1 - |f|^2) \leq w(f)$.

To prove the reverse inequality, given $\varepsilon > 0$, use (22) to find a polynomial $P(z) = 1 + b_1 z + \ldots + b_k z^k$ such that

$$(32) \quad \mathcal{A}(|P|^2 (1 - |f|^2)) \leq \mathcal{L}(1 - |f|^2) + \varepsilon .$$

If we adopt the convention that $u_m = 0$ for $m < 0$, then

$$P(z)f(z) = \sum_{m=0}^{\infty} \left(\sum_{j=0}^{k} b_j u_{m-j} \right) z^m .$$

Thus, applying Parseval's relation, the left member of (32) can be written

$$(33)\qquad \mathcal{A}(|P|^2 - |Pf|^2) = \mathcal{A}(|P|^2) - \sum_{m=0}^{\infty} \left| \sum_{j=0}^{k} b_j u_{m-j} \right|^2 .$$

Now A_n/Q_n is holomorphic, and we have

$$(34)\qquad \omega_n = \mathcal{L}(1 - |A_n/Q_n|^2) \leq \mathcal{A}(|P|^2 - |PA_n/Q_n|^2)$$

$$\leq \mathcal{A}(|P|^2) - \sum_{m=0}^{n} \left| \sum_{j=0}^{k} b_j u_{m-j} \right|^2 ,$$

by Bessel's inequality and (11). Let $n \to \infty$ in (34), then use (32) and (33) to obtain

$$w(f) \leq \mathcal{A}(|P|^2 - |Pf|^2) \leq \mathcal{L}(1 - |f|^2) + \varepsilon .$$

Since $\varepsilon > 0$ is arbitrary, we have proved (23) for $p = 0$ and hence for all p.

The proof of (24) uses $|\Omega| = |Q|^2(1 - |f|^2)$ on $|z| = 1$, and (21) to identify $\mathcal{L}(|Q|^2) = |Q(0)|^2 \theta_1^2 \cdots \theta_p^2$. The proof of (25) uses (21) and the fact that Ω is a reciprocal polynomial.

Remarks. 1. For $f \in H^\infty$, the condition for f to be an extreme point of the unit ball is that (in addition to $|f(e^{it})| \leq 1$ a.e.), $\mathcal{G}(1 - |f|) = 0$ [11,p.138]. This is the same as $\mathcal{L}(1 - |f|^2) = 0$ since $1 \leq \mathcal{L}(1 + |f|) \leq 2$. Thus $w(f) = 0$ for $f \in \mathcal{M}_0$ if and only if f is an extreme point of the unit ball of H^∞. This is not an altogether unexpected result.

2. An earlier proof which we obtained for Theorem 1 in case of rational $f = A/Q$ contains a result of some independent interest. Namely, the sequences A_n, Q_n individually converge uniformly on compact subsets of the unit disk to rational functions A/F, Q/F where F has all its zeros in $|z| \geq 1$

and $|Q|^2 - |A|^2 = w(f)\,|F|^2$ on $|z| = 1$. The polynomials A_n^* and Q_n^* converge to zero in the unit disk. Note that A/F and Q/F are holomorphic in $|z| < 1$. This convergence result is much stronger than the fact that $A_n/Q_n \to A/Q$ uniformly in any disk $|z| \leq \delta$ free of zeros of Q . Since this result is not particularly relevant here, we will present it elsewhere, along with a generalization to the full class $\mathcal{M}_p$.

Corollary 1. If $f \in \mathcal{C}$, then $w(f) = 0$ or $w(f) \geq 1$. These correspond respectively to $h(f) = 0$ or $h(f) \geq 1$.

Proof: Since $\Omega(0)$ is an integer, the first statement is immediate from (25). The second statement follows from the characterization of $\mathcal{C}^{(1)}$, [4].

Corollary 2. If $f \in \mathcal{M}_1$ with pole at θ^{-1}, then

$$(35) \qquad w(f) \leq \theta^2 - \sum_{n=0}^{\infty} \left| u_n - (\theta - \theta^{-1}) \sum_{k=0}^{n-1} \theta^{-k} u_{n-k-1} \right|^2 .$$

Proof: The function $g(z) = (1 - z\theta)f(z)/(\theta - z)$ is holomorphic in $|z| < 1$ and $|g(z)| = |f(z)|$ on $|z| = 1$. Hence,

$$w(f) = \theta^2 \mathcal{L}(1 - |f|^2) = \theta^2 \mathcal{L}(1 - |g|^2)$$

$$\leq \theta^2 \mathcal{A}(1 - |g|^2) = \theta^2 - \theta^2 \mathcal{A}(|g|^2) ,$$

using (20). The Taylor coefficients of g are

$$v_n = u_n \theta^{-1} - (1 - \theta^{-2}) \sum_{k=0}^{n-1} \theta^{-k} u_{n-k-1} .$$

Thus an application of Parseval's relation completes the proof.

Corollary 3. If $\theta \in S$, then $w(\theta) \leq \theta^2 - 1$.

Proof: If $f \in \mathcal{C}$ has its pole at θ^{-1} then (35) shows $w(f) \leq \theta^2 - u_0^2 \leq \theta^2 - 1$. Now apply Definition 3.

Corollary 4. $$\min S^{(h)} \geq (h+1)^{1/2} .$$

Proof: Let $\theta = \min S^{(h)}$. Then $h \leq w(\theta)$ by Lemma 1, hence $h \leq \theta^2 - 1$ by Corollary 3.

Corollary 5. If $\theta \in S$, then, for any integer $n \geq 1$,

$$w(\theta^n) \geq n - 1 + \theta^{2(n-1)} w(\theta) .$$

Proof: Let f have pole θ^{-1} and satisfy $w(\theta) = w(f)$. It may be shown, as in [4,p.114] that, for $j = 0,1,\ldots, n-1$, the function

$$f_j(z) = \sum_{m=0}^{\infty} u_{j+mn} z^m$$

is in $\mathcal{C}$, has a pole at $z = \theta^{-n}$, and that if ζ is a primitive nth root of unity:

$$\sum_{j=0}^{n-1} |f_j(z^n)|^2 = \frac{1}{n} \sum_{k=0}^{n-1} |f(\zeta^k z)|^2 .$$

Thus,

$$\begin{aligned}
w(f_0) &= \theta^{2n} \mathcal{L}(1 - |f_0|^2) = \theta^{2n} \mathcal{L}(1 - |f_0(z^n)|^2) \\
&= \theta^{2n} \mathcal{L}\left\{\sum_{j=1}^{n-1} |f_j(z^n)|^2 + n^{-1} \sum_{k=0}^{n-1} (1 - |f(\zeta^k z)|^2)\right\} \\
&\geq \theta^{2n} \left\{\sum_{j=1}^{n-1} \mathcal{L}(|f_j|^2) + n^{-1} \sum_{k=0}^{n-1} \mathcal{L}(1 - |f(\zeta^k z)|^2)\right\} \\
&\geq \theta^{2n} \left\{ \sum_{j=1}^{n-1} u_j^2 \theta^{-2n} + \mathcal{L}(1 - |f|^2) \right\} \\
&= \sum_{j=1}^{n-1} u_j^2 + \theta^{2n-2} w(f) .
\end{aligned}$$

We have used a few elementary changes of variable and $\mathcal{L}(|f_j|) \geq \theta^{-n}|f_j(0)|$, by Jensen's formula, ignoring any zeros f_j may have in $|z| < 1$. Since the u_j are non-zero integers, Corollary 5 is now proved. (Of course $u_j \sim \lambda\theta^j$ so for large j the result can be improved).

<u>Examples</u>. 1. Let $f = (1 - z^2)/(1 - kz - z^2)$. Then $\Omega(z) \equiv k^2$. Thus $w(f) = \mathcal{L}(|\Omega|) = k^2$. Note that $\theta = (k + (k^2+4)^{1/2})/2 \sim k$, showing that Corollary 1 is asymptotically sharp. Observe that $A_n = 1 - z$ and $Q_n = 1 - kz - z^2$ for $n \geq 2$, since we have seen that (12) is satisfied by this pair. Thus $\omega_n = k^2$ for $n \geq 2$.

2. Let $f = 1/(1 - kz)$. Then $\Omega(z) = -k\ (1 - kz + z^2)$. Hence $w(f) = \mathcal{L}(|\Omega|) = k(k + (k^2-4)^{1/2})/2 \sim k^2 - 1 - k^{-2}$. Thus $h(f) \leq [w(f)] = k^2 - 2$, which makes it conceivable that $f \in \mathcal{C}^{(k^2-2)}$ and $k \in S^{(k^2-2)}$. It is true that $2 \in S^{(2)}$ as shown in [9], but we do not know if $k \in S^{(k^2-2)}$ for any other value of k. This is discussed further in Section 5.

<u>Corollary 6</u>. If $k \geq 3$, then

$$w(k) = k(k + (k^2-4)^{1/2})/2 = w(1/(1 - kz)), \tag{36}$$

while

$$w(2) = (3 + \sqrt{5})/2 = w((1-z)/(1-2z)). \tag{37}$$

<u>Proof:</u> Let $b_k = k(k + (k^2-4)^{1/2})/2$. By Lemma 2, some $f \in \mathcal{C}$ has $w(f) = w(k)$. Let $f = u_0 + u_1 z + \ldots$ as usual. Then $u_0 = 1$, for if $u_0 > 1$, (35) shows that $w(f) \leq k^2 - 4 < b_k$. Again (35) gives

$$w(f) \leq k^2 - 1 - (u_1 - (k - k^{-1}))^2 \quad .$$

Thus if $u_1 \neq k$ we would have $w(f) \leq k^2 - 1 - (1 - k^{-1})^2 < b_k$, if $k \geq 3$. So, if $k \geq 3$ then $u_1 = k$. Suppose that we have shown that $w(f) \geq b_k$ implies $u_n = k^n$ for $n = 0,1, \ldots, m$. Then

$$u_{m+1} - u_m(\theta - \theta^{-1}) - \ldots - u_0\, \theta^{-m}(\theta - \theta^{-1})$$
$$= u_{m+1} - k^m(k - k^{-1}) - \ldots - k^{-m}(k - k^{-1}) = u_{m+1} - k^{m+1} + k^{-(m+1)}.$$

If $u_{m+1} \neq k^{m+1}$ then (35) implies $w(f) \leq k^2 - 1 - (1 - k^{-(m+1)})^2 < b_k$. Hence $u_{m+1} = k^{m+1}$. Thus, by induction, if $k \geq 3$, then $u_n = k^n$ for all n if $w(f) \geq b_k$. Thus $f = 1/(1 - kz)$ and $w(f) = b_k$.

The argument breaks down for $k = 2$, since $u_1 = 1$ is equally suitable, but a similar proof will give (37).

4. The width functional applied to sequences of functions.

We now return to the study of $\mathcal{C}$. We first observe that w is far from continuous on $\mathcal{C}$. To the contrary, we have

Lemma 4. If $f_n \to f$ in $\mathcal{C}$ and $f_n \neq f$ for any n, then

$$\limsup w(f_n) \leq w(f) - 1/w(f) \quad .$$

Proof: See the proof of Lemma 2, especially (6).

The following is obvious:

Lemma 5. If $f_n \to f$ in $\mathcal{C}$ and $f_n \neq f$ for any n, then

$$\limsup h(f_n) \leq h(f) - 1.$$

We shall be interested in determining $\lim w(f_n)$ for specific sequences f_n. We begin with some examples to motivate the next theorem:

1. Consider $f = 1/(1 - 3z)$ with $w(f) = 3(3 + \sqrt{5})/2$ as we have seen. One readily verifies the following equation for $|z| = 1$,

$$|1 - 3z + z^n|^2 = 1 + |1 - z|^2 + |1 - z^{n-2}|^2 + |1 - z^{n-1}|^2 + |1 - z|^2 \, |1 - z^{n-1}|^2 .$$

Thus, for example $f_n = (1 - z^{n-2})/(1 - 3z + z^n) \in \mathcal{C}^{(1)}$, and $f_n \to f$. We calculate

$$z^n((1 - 3z + z^n)(1 - 3z^{-1} + z^{-n}) - (1 - z^{n-2})(1 - z^{-(n-2)}))$$
$$= (1 - 3z + z^2)(1 - 3z^{n-1} + z^{2n-2}) ,$$

and hence from Theorem 1, and Jensen's formula that

$$w(f_n) = \mathcal{L}(|1 - 3z + z^2|)\,\mathcal{L}(|1 - 3z^{n-1} + z^{2n-2}|)$$
$$= ((3 + \sqrt{5})/2)^2 = w(f) - 1 .$$

2. Before this last example leads us to seek an improvement of Lemma 4, let us consider $f_n = 1/(1 - 3z + z^n)$. For this sequence, we have

$$\Omega_n = z^n(1 - 3z + z^n)(1 - 3z^{-1} + z^{-n}) - z^n \tag{38}$$
$$= z^{2n}(1 - 3z^{-1}) + z^n(1 - 3z)(1 - 3z^{-1}) + (1 - 3z) .$$

A numerical calculation of the zeros of Ω_n shows that, for $n = 2,\ldots,7$, $w(f_n) = 3.732$, 5.552, 6.457, 6.873, 7.079, 7.189. This suggests that $w(f_n)$ increases to a limit greater than 7. To obtain a heuristic idea about the value of $w(f_n)$ we observe that Ω_n has a root ϕ_n with limit 3 as $n \to \infty$, as is easily proved. To see how the other roots of Ω_n in $|z| > 1$ may be distributed, let ω satisfy $\omega^{n-1} = 1$. Writing $z = \zeta^{1/n}\omega$, we find that

$$\Omega_n(z) = \zeta^2 \omega^2 (1 - 3z^{-1}) - \zeta \omega (1 - 3z)(1 - 3z^{-1}) + (1 - 3z) .$$

This strongly suggests that Ω_n should have a root near $\zeta^{1/n}\omega$, where ζ now satisfies

$$(39) \qquad \zeta^2\omega(1 - 3\,\overline{\omega}) - \zeta(1 - 3\omega)(1 - 3\,\overline{\omega}) + \overline{\omega}\,(1 - 3\omega) = 0 .$$

Now, for any fixed $\omega = e^{it}$, (39) indeed has a pair of roots $\zeta_1(t)$ and $\zeta_2(t)$ with $|\zeta_1(t)| > 1 > |\zeta_2(t)|$. If the n-1 roots of Ω_n in $|z| > 1$ other than ϕ_n are indeed sufficiently well approximated by $\zeta_1((2k\pi)/(n-1))^{1/n} \exp((2k\pi i)/(n-1))$ for $k = 0,1, \ldots, n-2$, we would have

$$\log w(f_n) \approx \log \phi_n + n^{-1} \sum_{k=0}^{n-2} \log |\zeta_1((2k\pi)/(n-1))|$$

$$\to \log 3 + (2\pi)^{-1} \int_0^{2\pi} \log |\zeta_1(t)|\, dt \quad , \text{as} \quad n \to \infty ,$$

Thus, we might expect

$$(40) \qquad w(f_n) \to 3\,\mathcal{L}(|\zeta_1|) .$$

We shall give a correct proof of (40), following slightly different lines, which relies on the following result:

<u>Theorem 2</u>. Let $P(\zeta,z)$ be a polynomial in two variables with complex coefficients. Then, writing $z = e^{it}$, $\zeta = e^{i\tau}$, we have

$$(41) \qquad \lim_{n\to\infty} \int_0^{2\pi} \log |P(z^n,z)|dt = (2\pi)^{-1} \int_0^{2\pi} d\tau \int_0^{2\pi} \log |P(\zeta,z)|\, dt .$$

<u>Proof</u>: Write $P(\zeta,z) = a_0(\zeta)\, z^s + \ldots + a_s(\zeta) = a_0(\zeta)(z - z_1(\zeta))\cdots(z - z_s(\zeta))$, where the z_j are the branches of the algebraic function defined by $P(\zeta,z) = 0$. We have then

$$(42)\qquad \int_0^{2\pi} \log |P(z^n,z)|\, dt = \int_0^{2\pi} \log |a_0(z^n)|\, dt + \sum_{j=1}^{s} \int_0^{2\pi} \log |z - z_j(z^n)|\, dt \,,$$

whereas, by Jensen's formula,

$$(43)\qquad \int_0^{2\pi} \log |P(\zeta,z)|\, dt = 2\pi \left\{ \log |a_0(\zeta)| + \sum_{j=1}^{s} \log^+ |z_j(\zeta)| \right\} .$$

Comparing (41),(42) and (43), we see that it suffices to show that

$$(44)\qquad \lim_{n\to\infty} \int_0^{2\pi} \log |a_0(z^n)|\, dt = \int_0^{2\pi} \log |a_0(\zeta)|\, d\tau \,,$$

and

$$(45)\qquad \lim_{n\to\infty} \int_0^{2\pi} \log |z - z_j(z^n)|\, dt = \int_0^{2\pi} \log^+ |z_j(\zeta)|\, d\tau \,.$$

Equation (44) follows from the change of variable $\tau = nt$. To prove (45), let us, for convenience, write $g(\zeta) = z_j(\zeta)$ and then

$$(46)\qquad \int_0^{2\pi} \log |e^{it} - g(e^{int})|\, dt = n^{-1} \int_0^{2\pi n} \log |e^{i\tau/n} - g(e^{i\tau})|\, d\tau$$

$$= \int_0^{2\pi} \left\{ n^{-1} \sum_{k=0}^{n-1} \log |\exp i((\tau + 2k\pi)/n) - g(e^{i\tau})| \right\} d\tau \,.$$

Our result is thus reduced to justifying the interchange of limit and integral in (46) since the integrand in the right member of (46) is a Riemann sum approximating an integral whose value is the integrand in the right member of (45), as one sees by applying Jensen's formula. To justify this interchange, we shall expand $\log|e^{it} - g(e^{i\tau})|$ into a Fourier series in e^{it} so that

$$(47) \qquad \log|e^{it} - g(e^{i\tau})| = \sum_{m=-\infty}^{\infty} c_m(\tau)\, e^{imt} \quad ,$$

where, as we see from the Taylor expansion of $\log(1-w)$, $c_0(\tau) = \log^+|g(e^{i\tau})|$, and

$$(48) \qquad c_m(\tau) = -|m|^{-1}\, g(e^{i\tau})^{\pm m} \quad \text{or} \quad -|m|^{-1}\, \overline{g(e^{i\tau})}^{\,\pm m} \quad ,$$

for $m \neq 0$, where the sign is such that $|c_m(\tau)| = \exp\{-|m|\, c_0(\tau)\}$.

We then find that

$$(49) \qquad n^{-1} \sum_{k=1}^{n} \log|\exp i((\tau + 2k\pi)/n) - g(e^{i\tau})|$$

$$= c_0(\tau) + \sum_{m\neq 0} c_{mn}(\tau)\, e^{im\tau} \quad ,$$

so that the desired interchange will be justified if

$$(50) \qquad \lim_{n\to\infty} \sum_{m\neq 0} \left| \int_0^{2\pi} c_{mn}(\tau)\, e^{im\tau}\, d\tau \right| = 0 \quad .$$

To prove (50), we use the fact that $g(\zeta)$ is a branch of an algebraic function and hence in the neighbourhood of any point ζ_0 it has a convergent expansion $g(\zeta) = g(\zeta_0) + a(\zeta - \zeta_0)^{\alpha} + \ldots$, where α is a rational number. We can divide $[0,2\pi)$ into a finite number of subintervals I so that in each we have $|g| \leq 1$ or $|g| \geq 1$ with equality at one endpoint only, or else $|g| = 1$ throughout I. We can further assume that either $g' \neq 0$ throughout I or else $g' = 0$ at one endpoint only. We thus can write

$$(51) \qquad \int_0^{2\pi} g(e^{i\tau})^{\pm mn}\, e^{im\tau}\, d\tau$$

as a sum of integrals over the subintervals. The contributions to (51) of each term of this sum is one of the following:

(i) $O(b^{|mn|})$ with $b < 1$, if $|g| \geq b^{-1}$ or $|g| \leq b$ in I,

(ii) If $|g| = 1$ at one endpoint only, change variables so this point becomes $\tau = 0$, and we have a contribution of the sort

$$\int_0^{\varepsilon} (1 - c\tau^{\alpha} + o(\tau^{\alpha}))^{|mn|} \, d\tau = O(|mn|^{-1/\alpha}) \quad .$$

(iii) If $|g| = 1$ throughout I and $g' \neq 0$ at any point in I, note that $g(e^{i\tau}) = e^{iu(\tau)}$ where u is real-valued and $u' \neq 0$. Integrating by parts shows such an interval contributes $O(|mn|^{-1})$.

(iv) If $|g| = 1$ throughout I and $g' = 0$ at an endpoint, change variables to obtain an integral of the type

$$\int_0^{\varepsilon} e^{inmu(\tau)} e^{im\tau} \, d\tau \quad ,$$

where $u(\tau) = c\,\tau^{\alpha} + o(\tau^{\alpha})$ as $\tau \to 0+$, and $\alpha > 1$. The integral over $[0,|nm|^{-1/\alpha}]$ is $O(|nm|^{-1/\alpha})$ and the remaining integral can be treated as in (iii) to obtain an estimate $O(\max(|nm|^{-1/\alpha}, |m|^{-1}))$. (Compare the discussion of the "method of stationary phase" in [8,pp.51-56]).

Combining (i)-(iv), we find that the sum in (50) is $O(n^{-2})$ if $g'(\zeta)$ does not vanish at a point where $|g(\zeta)| = 1$, and is $O(n^{-1-c})$ for some $0 < c < 1$ otherwise. This completes the proof.

Remark. The above proof is admittedly quite complicated, and it is reasonable to ask for a simpler proof under more general conditions. The following example may be instructive: let $G(e^{it}) = \exp(2^k it)$ for $(2 - 2^{2-k})\pi \leq t < (2 - 2^{1-k})\pi$, for $k = 1,2,\ldots$. Then G is continuous on the circle except at $t = 2\pi$. Let $F(e^{i\tau},e^{it}) = g(e^{it}) - e^{i\tau}$. Then

$$\int_0^{2\pi} \log |F(e^{int}, e^{it})| \, dt = -\infty \qquad \text{for} \quad n = 2^k, \; k = 1,2,\ldots ,$$

but

$$\int_0^{2\pi} dt \int_0^{2\pi} \log |F(e^{i\tau}, e^{it})| \, d\tau = 0 ,$$

by Jensen's theorem applied to the inner integral.

Corollary 7. Let $A/Q \in \mathcal{C}^{(2)}$ and let B,C be polynomials such that $|A(z) + z^n B(z)| \le |Q(z) + z^n C(z)|$ for $|z| = 1$ and all $n \ge 0$, with equality at a finite number of points at most, in each case. Let

$$K(z) = C(z)Q(z^{-1}) - B(z)A(z^{-1})$$

$$L(z) = Q(z)Q(z^{-1}) - A(z)A(z^{-1}) + C(z)C(z^{-1}) - B(z)B(z^{-1})$$

$$N(z) = (L + (L^2 - 4|K|^2)^{1/2})/2 .$$

If $f_n = (A + z^n B)/(Q + z^n C)$, then

$$\lim_{n \to \infty} w(f_n) = \mathcal{L}(N) . \tag{52}$$

Proof: Write $A^o(z) = A(z^{-1})$ for polynomial A. From Theorem 1, $w(f_n)$ is the geometric mean of $\Omega_n = z^r \Gamma_n$, where

$$\Gamma_n(z) = (Q + z^n C)(Q^o + z^{-n} C^o) - (A + z^n B)(A^o + z^{-n} B^o)$$

$$= z^n K + L + z^{-n} K^o .$$

Since $|Q + z^n C| \ge |A + z^n B|$ on $|z| = 1$, we know that $\Gamma_n \ge 0$ on $|z| = 1$. If $z = \exp(2\pi i \alpha)$ with α irrational we can choose n so that $|z^n K(z) - |K(z)|| < \varepsilon$ and hence we must have $L \ge 2|K|$ for such z, hence for all $|z| = 1$, by continuity. According to Theorem 2, and Fubini's theorem,

$$\lim_{n \to \infty} \mathfrak{L}(\Omega_n) = \exp\left\{ \frac{1}{2\pi}\int_0^{2\pi} dt \ \frac{1}{2\pi}\int_0^{2\pi} \log |\zeta K + L + \zeta^{-1}K^o| \, d\tau \right\} .$$

Evaluating the inner integral by Jensen's formula, we obtain (52), since the above discussion shows that the modulus of the larger zero of $\zeta^2 K + \zeta L + K^o = 0$ is $N/|K|$.

Examples: 1. Returning to the example $f_n = 1/(1 - 3z + z^n)$ we have $K(z) = 1 - 3z^{-1}$, $L(z) = (1 - 3z)(1 - 3z^{-1})$ and hence, on $|z| = 1$, $L^2 - 4|K|^2 = L^2 - 4L = 3L(1-z)(1-z^{-1})$. Using (19), we can estimate

$$\mathfrak{L}\{(L + (L^2 - 4|K|^2)^{1/2})/2\} \geq \mathfrak{L}(L)/2 + \mathfrak{L}(3L(1-z)(1-z^{-1}))^{1/2}/2$$
$$= (9 + 3\sqrt{3})/2 > 7.098 .$$

Thus $\lim_{n \to \infty} w(f_n) > 7$ and hence $w(f_n) > 7$ except for a finite set of values. However, Lemma 5 shows that $\limsup h(f_n) \leq h(f) - 1 \leq 6$. Thus, it is not true in general that $h(f) = [w(f)]$ for $f \in \mathcal{C}$, as one might have hoped.

2. We have seen that $w(2) = w((1-z)/(1-2z)) = (3 + \sqrt{5})/2$ and thus that $w(2) - 1/w(2) = \sqrt{5} > 2$. Lemma 4 thus does not rule out the possibility that $\lim w(f_n) > 2$ for some sequence f_n tending to $(1-z)/(1-2z)$ so possibly there might be $\theta < 2$ with $w(\theta) \geq 2$. We shall show that this is not the case. Amara [1] has determined all $\theta \in S^{(1)}$ with $\theta < 2$ and all associated $f \in \mathcal{C}^{(1)}$, so it is only necessary to examine these f, which are:

$$f_0 = (1 + z - z^2 - z^3)/(1 - z - 2z^2 + z^4) \quad ,$$

$$f_{1,n} = (1 - z^{n-2})/(1 - 2z + z^n) \quad , \quad n \geq 3 \quad ,$$

$$f_{2,n} = (1 - z)(1 - z^{n-1})/(1 - 2z + z^n) \quad , \quad n \geq 3 \quad ,$$

$$g_{1,n} = (1 - z + z^n)/(1 - 2z + z^n - z^{n+1}) \quad , \quad n \geq 2 \quad ,$$

$$g_{2,n} = (1 - z^{n-1} + z^n)/(1 - 2z + z^n - z^{n+1}) \quad , \quad n \geq 2 \quad ,$$

$$g_{3,n} = (1 - z)(1 + z^n)/(1 - 2z + z^n - z^{n+1}) \quad , \quad n \geq 2 \quad .$$

We find, using Theorem 1, that

$$w(f_0) = \mathcal{L}(|1 - z^2 + z^4|^2) = 1$$

$$w(f_{1,n}) = \mathcal{L}(|1 - z|^2|1 - z^{n-1}|^2) = 1$$

$$w(f_{2,n}) = \mathcal{L}(|1 - z^{n-2}|^2) = 1 \quad ,$$

and

$$w(g_{1,n}) = w(g_{2,n}) = \mathcal{L}(|1 - z|^2|1 - z^{n-1}|^2) = 1 \quad ,$$

leaving only $g_{3,n}$ for which we find

$$\Omega_n = z^{2n}(1 - z^{-1}) - z^n(z - 3 + z^{-1}) + (1 - z) \quad ,$$

and hence that $N(e^{it})$ in Corollary 5 is given by

$$N(e^{it}) = \begin{cases} 1/2 \sin(t/2) & \text{if} \quad |t| \leq \pi/3 \\ 2 \sin(t/2) & \text{if} \quad \pi/3 \leq t \leq \pi \end{cases} \quad .$$

Thus, we have

$$\lim_{n \to \infty} w(g_{3,n}) = \mathcal{L}(N) = \exp\left\{\frac{2}{\pi}\int_0^{\pi/3} - \log(2 \sin(t/2))\, dt\right\} \quad ,$$

where we have used the well known fact that $\int_0^{\pi} \log(2 \sin(t/2))\, dt = 0$ to show that the integrals of $\log N$ over $[0,\pi/3]$ and $[\pi/3,\pi]$ are equal. The integral can be evaluated by setting $u = 2 \sin(t/2)$ to obtain

$$\int_0^1 (-\log u)\,(1-u^2/4)^{-1/2}\,du = \sum_{m=0}^{\infty}\binom{-1/2}{m}(-1/4)^m \int_0^1 (-\log u)\,u^{2m}\,du$$

$$= \sum_{m=0}^{\infty}\binom{-1/2}{m}(-1/4)^m (2m+1)^{-2} = 1.014941606 \quad ,$$

which gives $\mathfrak{L}(N) = 1.908145627$. The fact that $\mathfrak{L}(N) < 2$ could also be seen from $1 - u^2/4 > 1 - u^2$ and $\int_0^1 (-\log u)(1-u^2)^{-1/2}du = (\pi \log 2)/2$.

Thus, at most a finite number of $w(g_{3,n})$ are > 2. By making careful estimates as in the proof of Theorem 2, we shall show that no such n exist. For, we can factor $\Omega_n = (z^n - z + 1)(z^n - z^{n-1} + 1)$ and hence we have $\mathfrak{L}(|\Omega_n|) = \mathfrak{L}(|z^n - z + 1|)^2$. Now we can write

$$\log |e^{i\tau} - e^{it} + 1| = \sum_{m=-\infty}^{\infty} c_m(\tau)\,e^{imt} \quad ,$$

where , for $m \neq 0$,

$$|c_m(\tau)| = \begin{cases} (2|m|)^{-1}(2\cos(\tau/2))^{-|m|} & \text{if } |\tau| \le 2\pi/3 \quad , \\ (2|m|)^{-1}(2\cos(\tau/2))^{|m|} & \text{if } 2\pi/3 \le |\tau| \le \pi \quad . \end{cases}$$

As in (49), we have

$$\left| \log \mathfrak{L}(|z^n - z + 1|) - (2\pi)^{-1}\int_0^{2\pi} \log^+ |1 + e^{i\tau}|\,d\tau \right|$$

$$\le \sum_{m\neq 0} (2\pi)^{-1}\int_{-\pi}^{\pi} |c_{mn}(\tau)|\,d\tau \quad .$$

We have the following estimates, taking $m > 0$:

$$\int_{2\pi/3}^{\pi} (2\cos(\tau/2))^m\,d\tau = \int_0^1 u^m\,(1-u^2/4)^{-1/2}\,du$$

$$\le (2/\sqrt{3})\int_0^1 u^m\,du < (2/\sqrt{3})\,m^{-1} \quad .$$

Also

$$(54)\qquad \int_0^{2\pi/3} (2\cos(\tau/2))^{-m}\, d\tau = \int_0^1 (1+u)^{-m} \left\{1 - (1+u)^2/4\right\}^{-1/2} du \ .$$

If we divide the integral on the right into two integrals over $[0,c]$, $[c,1]$, where $c = (2m)^{1/m} - 1$, and estimate each in the obvious manner, we find that the integral in (54) is bounded by

$$2\, m^{-1} (4 - (2m)^{2/m})^{-1/2} \left\{(2m)^{1/m} (1 - 1/2m) + 2 - (2m)^{1/m}\right\}$$

$$\leq 4\, m^{-1} (4 - (2m)^{2/m})^{-1/2} < 2.7095\, m^{-1} \ , \quad \text{if} \quad m \geq 10 \ .$$

Thus, we obtain

$$(2\pi)^{-1} \int_{-\pi}^{\pi} |c_m(\tau)|\, d\tau \leq .6150\, m^{-2} \quad \text{for } |m| \geq 10 \ ,$$

and hence that the error in (53) is bounded by

$$\sum_{m\neq 0} .6150\, m^{-2} n^{-2} = (\pi^2/3)(.6150)\, n^{-2} = 2.0233\, n^{-2} \quad \text{for} \quad n \geq 10.$$

This suffices to show that $w(g_{3,n}) < 2$ for $n \geq 10$, and in fact to show that

$$\max_n \ w(g_{3,n}) = w(g_{3,5}) = 1.987738267 \ ,$$

so that $w(\theta)$ for $\theta < 2$ attains a maximum for $\theta = 1.967168213$, the largest root of $1 - 2z + z^5 - z^6$ with

$$w(\theta) = w((1 - z + z^5 - z^6)/(1 - 2z + z^5 - z^6)) \ .$$

6. <u>The derived sets of S</u>.

According to Corollary 6 and Lemma 1, if θ is the rational integer k, we have $h(k) \leq [w(k)] = k^2 - 2$. The estimate $h(k) \geq k - 1$ [13,p.57] follows by applying Rouche's theorem to the polynomial

$$1 - kz + z^{n_1} + \dots + z^{n_{k-1}} \quad .$$

This has a single root in $|z| < 1$, say $1/\theta_{n_1,\dots,n_{k-1}}$ which depends on $k - 1$ parameters, each of which can tend to ∞ independently, the ultimate limit being k^{-1}. We suspect that $h(k) \sim k^2$ is correct, but have been unable to show this. We do have the following which at least shows that the most pessimistic guess, $h(k) = k$, is not correct for large k.

<u>Theorem 3</u>. For $\theta = k$ a rational integer,

$$h(k) \geq \max\left(2[\alpha(k-1)],\ 2[\alpha(k-2)] + 1 \right) , \tag{55}$$

where $\alpha = 1/(2\sqrt{2} - 1) > 1/2$. In particular ,

$$\liminf\, (\min S^{(h)}/h) \geq 2\alpha > 1 .$$

<u>Proof</u>: Let $P(a,b,c;z) = z^a + z^b + z^{a+c} - z^{b+c}$, for integer a,b,c . Then, for $|z| = 1$,

$$|P(a,b,c;z)|^2 = |z^a + z^b|^2 + (z^c + z^{-c})(z^{a-b} - z^{b-a}) + |z^a - z^b|^2 \leq 8,$$

combining the first and last terms and using the parallelogram law. Thus, if $\ell - 1 \geq 2\sqrt{2}\, m$ and $n_1, \dots , n_{3m}$ are arbitrary positive integers, then the following polynomial has exactly one zero in $|z| < 1$:

$$1 - \ell z + P(n_1,n_2,n_3;z) + \dots + P(n_{3m-2},n_{3m-1},n_{3m};z) \quad .$$

This shows that $\ell \in S^{(3m)}$ since we may let $n_1 ,\dots, n_{3m}$ tend to ∞ in a suitable order, (i.e. n_1 , n_2 may not both be allowed to tend to ∞ until we have let $n_3 \to \infty$ or else we "lose" the parameter n_3).

We can specialize $n_{3i+1} = 1$ for $1 \leq i \leq j$, where $j \leq m$ and obtain a polynomial which shows that $k = \ell - j \in S^{(3m-j)}$ if $\ell - 1 \geq 2\sqrt{2}\, m$.

The most favourable choice of j is $j = m$, giving $k \in S^{(2m)}$ if $k - 1 \geq (2\sqrt{2} - 1)m$, or else $j = m - 1$, giving $k \in S^{(2m+1)}$ if $k - 2 \geq (2\sqrt{2} - 1)m$. These combine to give (55).

Remarks. 1. For $3 \leq k \leq 11$, Theorem 4 gives only $h(k) \geq k - 1$, but for $12 \leq k \leq 22$ it gives $h(k) \geq k$ and for larger k we have $h(k) > k$, e.g. $h(23) \geq 24$.

2. There is a possibility of obtaining improved upper bounds on $h(\theta)$ in the following way: Grandet-Hugot [9,p.24] states that if $\theta \in S^{(h)}$ and $A/Q \in \mathcal{C}^{(h)}$ is the associated rational function, and if $\theta_n \in S$ is associated with the polynomial $Q(z) + z^n A(z)$, then $\theta_n \in S^{(h-1)}$. Since $2 \notin S^{(3)}$ is associated with $1 - 2z = (1 - 3z) + z(1)$, this would appear to show that $h(1/(1-3z)) \leq 3$. However, the result in question may fail for a finite set of n corresponding to values for which $z^n B_I + C_I \equiv 0$, for certain I, in the notation of [9]. Since we do not know a priori whether n is one of these values, we cannot draw the desired conclusion.

3. We conclude by showing that $h(2/(1-3z)) = w(2/(1-3z)) = 3$, which shows that $h(3) \geq 3$. We calculate $\Omega = -3(1-z)^2$ so that $w(f) = 3$ by Theorem 1. To prove that $f \in \mathcal{C}^{(3)}$ it suffices to show that $f_{m,n} \in \mathcal{C}^{(1)}$, where

$$(56) \qquad f_{m,n}(z) = \frac{2 - z^n(1+z) + z^m(1-z) - z^{m+n+1}(1-z)}{1 - 3z + 2z^{n+2} - z^{m+1}(1-z) + z^{m+n+2}(1-z)} .$$

This can be verified directly, but we shall show how it was derived. We begin by seeking C/D which differ from A/Q in the nth coefficient. According to [9,pp.9-12], these all have the form

$$C/D = (AU + z^{n+1}Q^{o}V)/(QU + z^{n+1}A^{o}V) ,$$

where V,U are polynomials with integer coefficients which satisfy $|V/U| \leq 1$ on $|z| = 1$, and we have written $A^{o}(z) = A(z^{-1})$, $Q^{o}(z) = Q(z^{-1})$. The polynomials C,D may have a common factor, but this must be a factor of Ω. The easiest case to handle is V/U holomorphic in $|z| \leq 1$, since this automatically ensures the correct number of zeros of D in $|z| < 1$. We must have $U(0) \mid \Omega(0)$, and $U(0) = 1$ is easily seen to be of no value here, so we try $U(0) = 3$. We find that the choice $U = 3 - 2z^{n+1}$, $V \equiv 1$ makes $(1 - 3z)U + 2z^{n+1}V$ divisible by 3. Letting $C_1 = C/3$ and $D_1 = D/3$ we have $C_1 = 2 - z^n - z^{n+1}$, $D_1 = 1 - 3z + 2z^{n+2}$ and $\Omega(C_1,D_1) = 2(1-z)^2(1 - z^{n+1})^2$, using an obvious notation. Hence $w(C_1,D_1) = 2$. Observe that $C_1(1) = D_1(1) = 0$; however it is unnecessary to divide by $(1 - z)$, and it avoids complications if we do not.

Now we seek $E/F = (U_1C_1 + z^{m+n+2}D_1^{o}V_1)/(U_1D_1 + z^{m+n+2}C_1^{o}V_1)$, where any common factor of E and F divides $\Omega(C_1,D_1)$. The choice $U_1 = 2 - z^{m+1}$, $V_1 \equiv 1$ makes $E \equiv F \equiv 0 \pmod 2$. Dividing E and F by the common factor $2(1-z)$ produces $f_{m,n} = C_2/D_2$ of (56) with $\Omega(C_2,D_2) = (1 - z^{n+1})^2(1 - z^{m+1})^2$.

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